Problem 1. Let $\mathcal{V} = (-1, 1)$. Define the addition and the scalar multiplication in $\mathcal{V}$ by: For all $u, v \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$ set

$$u \diamondplus v = \frac{u+v}{1+uv}, \qquad \alpha \diamonddot v = \frac{(1+v)^\alpha - (1-v)^\alpha}{(1+v)^\alpha + (1-v)^\alpha}.$$

Prove that $\mathcal{V}$ with the vector addition $\diamondplus$ and the scaling $\diamonddot$ is a vector space.

The vector space in the above problem is a strange vector space. It is an excellent exercise to go through all ten axioms of the vector space and see whether each is satisfied. To prove that all ten axioms are satisfied, you will have to prove two inequalities and eight algebraic identities. Some of the proofs involve several steps. You can at least state what is needed to be proved for each axiom and try to prove some of the statements.

Problem 2. Consider the vector space $\mathcal{V}$ of all real valued functions defined on $\mathbb{R}$, see Example 5 on page 194. The purpose of this exercise is to study a special subspace of the vector space $\mathcal{V}$. Consider the set

$$\mathcal{S}_1 := \Big\{ f \in \mathcal{V} : \exists\, a, b \in \mathbb{R} \ \text{ such that } \ f(x) = a \sin(x+b) \ \ \forall\, x \in \mathbb{R} \Big\}.$$

Prove that $\mathcal{S}_1$ is a subspace and determine its dimension.

The magic tool to prove that the set given in the above problem is a subspace is the concept of a span. In fact, you simply prove that the given set equals a span of two functions.